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An exact solution for the impact law in thick elastic plates ☆

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Abstract

We study the impact of a rigid sphere on a circular elastic plate whose thickness is not small with respect to its diameter, so that Kirchhoff's theory cannot be applied. For plate-like bodies of this kind it is convenient to apply a theory proposed by Levinson [J. Elasticity 7 (1985) 283], which is a compromise between Kirchhoff's solution and that obtained by the integration of Lamé's equation of three-dimensional elasticity. The pressure distribution and the extent of the (circular) area of contact of the sphere on the plate-like body is mathematically described by Hertz's theory. By combining these two theories in a dynamical framework, we derive a non-linear ordinary differential equation able to describe the normal slow impact of a rigid sphere against an elastic plate-like body.
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Keywords: Elasticity theory; Impact law; Elastic plate; Dynamical impact

1. Introduction

In this paper we analyze the normal impact problem of a rigid sphere against a circular elastic plate. We assume that the radius of the sphere and the thickness of the plate are comparable and they are both not small with respect to the diameter of the plate. As a consequence, neither Kirchhoff's theory of thin plates nor Hertz's theory of the elastic semi-space can be applied.

The problem is approached by searching a suitable solution of the equations of the three-dimensional theory of elasticity in a cylindrical domain (Villaggio, 1997). To this end, we exploit the work of Levinson (1985) which proposed a static theory that is a compromise between Kirchhoff's plate solution and that obtained by the integration of Lamé's equation of three-dimensional elasticity. In this paper we propose an extension of the Levinson's displacement field in the dynamical case.

In Section 2 we consider a circular isotropic elastic plate with axisymmetric load conditions and write the elastodynamic equations. We adopt a semi-inverse method by assuming detailed load conditions on the plate faces and satisfying boundary conditions only partially on the mantle. The remaining boundary

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conditions may be considered as compatibility conditions on the data for the applicability of the model. We propose a solving technique based on the method of the separation of the variables and the use of the Fourier–Bessel expansion.

In Section 3 we assign the detailed distribution of the pressure on the upper face of the plate and the extent of the (circular) area of contact by extending, to the dynamical case, Hertz's theory (Johnson, 1985) and, in Section 4, we explicitly obtain the impact law.

In Section 5 we study the equation of the motion for the rigid sphere impinging on the thick plate by assuming that the period of duration of the impact is much larger than the time employed by the elastic waves in traversing the plate after the first impact (Villaggio, 1996). According to this assumption, we adopt the static contact law and we get a non-linear second order ordinary differential equation for the dynamical value of the indentation. A perturbative technique allows us to get the numerical values of the contact period, the maximum contact force and the maximum indentation in terms of the plate thickness.

Finally, a numerical investigation is performed in order to valuate the influence of the thickness-to-side ratio on the impact law and on the analysis of the motion during a slow impact.

2. Axisymmetric Levinson-type problems

Let a circular plate-like body of thickness $2h$ and radius b referred to a system of cylindrical coordinates (r, θ, z) such that its origin 0 is placed at the center of the middle plane. By considering axisymmetric load conditions in a statical equilibrium problem, Levinson assumed the following expression for the displacement field

$$\begin{aligned} u(r,z) &= -g(z)\frac{\mathrm{d}}{\mathrm{d}r}W(r), \\ w(r,z) &= f(z)W(r). \end{aligned} \tag{2.1}$$

The function $W(r)$ is the deflection of the middle surface (so that $f(0) = 1$) and $g(z)$ and $f(z)$ are functions determining the variations in the displacements through the thickness of the plate (Levinson, 1985).

In this paper, we consider a dynamical problem with axisymmetric load conditions and assume that the displacement field (2.1) may be written as follows

$$\begin{aligned} u(r,z,t) &= -g(z)\frac{\mathrm{d}}{\mathrm{d}r}W(r)\mathrm{e}^{\mathrm{i}\omega t}, \\ w(r,z,t) &= f(z)W(r)\mathrm{e}^{\mathrm{i}\omega t}. \end{aligned} \tag{2.2}$$

The boundary condition over the mantle $r = b$ is

$$w(b,z,t) = 0, \tag{2.3}$$

while, on the upper and lower faces, are

$$\begin{aligned} \sigma_{rz}(r,\pm h,t) &= 0, \\ \sigma_{zz}(r,-h,t) &= 0, \\ \sigma_{zz}(r,+h,t) &= p(r,t). \end{aligned} \tag{2.4}$$

The function $p(r,t)$ is the load on the upper face of the plate whose form will be detailed in the next section. We remark that the displacement boundary condition (2.3) is partly consistent with the notion of simple supported edge of classical bending theory.

By considering null body force, the linear elastodynamic equations for an isotropic material are

$$(\lambda + 2\mu)\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}u + \frac{u}{r} + \frac{\partial}{\partial z}w\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}u - \frac{\partial}{\partial r}w\right) - \rho\frac{\partial^2}{\partial t^2}u = 0, \tag{2.5}$$

$$(\lambda + 2\mu)\frac{\partial}{\partial z}\left(\frac{\partial}{\partial r}u + \frac{u}{r} + \frac{\partial}{\partial z}w\right) - \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial}{\partial z}u - \frac{\partial}{\partial r}w\right)\right) - \rho\frac{\partial^2}{\partial t^2}w = 0, \tag{2.6}$$

where ρ is the density and λ and μ the Lamé moduli.

By substituting the displacement field (2.2) in Eqs. (2.5) and (2.6), we get the following equations

$$\left(\mu g'' - (\lambda + \mu)f' - \frac{1}{r^2}(\lambda + 2\mu)g + \rho\omega^2 g\right)W' + (\lambda + 2\mu)\left(\frac{1}{r}W'' + W'''\right)g = 0, \tag{2.7}$$

$$((\lambda + 2\mu)f'' + \rho\omega^2 f)W + (\mu f - (\lambda + \mu)g')\left(\frac{1}{r}W' + W''\right) = 0. \tag{2.8}$$

The differential equation (2.8) can be transformed into the form

$$\frac{\Delta W}{W} = \frac{(\lambda + 2\mu)f'' + \rho\omega^2 f}{-\mu f + (\lambda + \mu)g'},$$

which allows variable separation; we write the left side as follows

$$\Delta W + \eta W = 0.$$

Since the boundary condition (2.3) implies $W(b) = 0$, a classical result ensures us that η is non-negative (see p. 298 of Courant and Hilbert (1953)). Therefore, by putting $\eta = k^2$, we get

$$\Delta W + k^2 W = 0, \tag{2.9}$$

$$(\lambda + 2\mu)f'' + \rho\omega^2 f + k^2(-\mu f + (\lambda + \mu)g') = 0. \tag{2.10}$$

The solution of Eq. (2.9) is

$$W(r) = BJ_0(kr) + DY_0(kr),$$

where $J_0(kr)$ and $Y_0(kr)$ are the 0-order Bessel functions of the first and second kind respectively. The coefficient D is zero since the function $Y_0(kr)$ is unbounded in $r = 0$, and hence function $Y_0(kr)$ must be disregarded. Further, the requirement $w(b, z, t) = 0$ implies $J_0(kb) = 0$ which is satisfied only if

$$W_j(r) = J_0(\phi_j r),$$

where $\phi_j = Z_j/b$ with Z_j the j–th positive zero of J_0.

Now we substitute this relation in (2.7) and differentiate it with respect to z. By using the expression of $g'(z)$ obtained by Eq. (2.10), we obtain

$$(\lambda + 2\mu)\mu f_j'''' - \left[2k_j^2\mu(\lambda + 2\mu) - \rho\omega^2(\lambda + 3\mu)\right]f_j'' + \left[k_j^4\mu(\lambda + 2\mu) - \rho\omega^2(\lambda + 3\mu)k_j^2 + \rho^2\omega^4\right]f_j = 0, \tag{2.11}$$

whose solution is

$$f_j(z) = C_1^{(j)}\cosh(\alpha_j z) + C_2^{(j)}\sinh(\alpha_j z) + C_3^{(j)}\cosh(\beta_j z) + C_4^{(j)}\sinh(\beta_j z), \tag{2.12}$$

where we use the index j to distinguish different solutions corresponding to different values of k_j.

The coefficients $C_1^{(j)}$, $C_2^{(j)}$, $C_3^{(j)}$ and $C_4^{(j)}$ (which depend on ω), are uniquely determined by the conditions on the faces of the body (2.4); the coefficients α_j and β_j are

$$\alpha_j = \sqrt{\phi_j^2 - \frac{\rho\omega^2}{\mu}}, \quad \beta_j = \sqrt{\phi_j^2 - \frac{\rho\omega^2}{\lambda + 2\mu}}.$$

Hence the functions $g_j(z)$, obtained from Eqs. (2.10) and (2.12), take the form

$$g_j(z) = -\frac{\alpha_j}{\phi_j^2}\left(C_1^{(j)}\sinh(\alpha_j z) + C_2^{(j)}\cosh(\alpha_j z)\right) - \frac{1}{\beta_j}\left(C_3^{(j)}\sinh(\beta_j z) + C_4^{(j)}\cosh(\beta_j z)\right). \tag{2.13}$$

The displacement field is obtained by considering the sum on all values of j, so obtaining the following expansions:

$$\begin{aligned} u(r,z,t) = -\sum_{j=1}^{\infty}\Bigg(&\frac{\alpha_j}{\phi_j}\left(C_1^{(j)}\sinh(\alpha_j z) + C_2^{(j)}\cosh(\alpha_j z)\right) \\ &+ \frac{\phi_j}{\beta_j}\left(C_3^{(j)}\sinh(\beta_j z) + C_4^{(j)}\cosh(\beta_j z)\right)\Bigg)J_1(\phi_j r)\mathrm{e}^{\mathrm{i}\omega t}, \\ w(r,z,t) = \sum_{j=1}^{\infty}\Big(&C_1^{(j)}\cosh(\alpha_j z) + C_2^{(j)}\sinh(\alpha_j z) + C_3^{(j)}\cosh(\beta_j z) \\ &+ C_4^{(j)}\sinh(\beta_j z)\Big)J_0(\phi_j r)\mathrm{e}^{\mathrm{i}\omega t}. \end{aligned} \tag{2.14}$$

The series for u and w are respectively a Dini and a Fourier–Bessel expansions. We assume that the displacement field is $\mathscr{C}^2$ for all $r \in [0, b]$. The theorems stating that this assumption yields the convergence and allows the term by term differentiability of these series can be found in Watson's book(1966) or obtained by a straightforward development of the results therein presented. [1]

3. Evaluation of the coefficients

We assume the following normal periodic Hertzian pressure acting on the upper face of the plate

$$p(r,t) = p(r)\mathrm{e}^{\mathrm{i}\omega t} = \frac{4\mu}{\pi(1-\nu)R}\sqrt{a^2 - r^2}\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{3.1}$$

where

$$a = \left(\frac{3R(2\mu + \lambda)}{16\mu(\mu + \lambda)}\right)^{\frac{1}{3}} P^{\frac{1}{3}}$$

is the contact area radius, R the radius of the rigid sphere and P the resultant pressure.

We observe that the periodic pressure assumption (3.1) may be also regarded as a single Fourier component of an impulsive impact load (of Dirac-like form), since the frequency in (3.1) is arbitrary.

The pressure $p(r)$ is written with a Fourier–Bessel expansion on the interval $(0, b)$

$$p(r) = \sum_{j=1}^{\infty} A_j J_0(\phi_j r), \tag{3.2}$$

[1] For details see the topics from proposition 18.24 up to proposition 18.4 in Watson (1966).

where the coefficients A_j are given by the formula

$$A_j = \frac{2\int_0^a rp(r)J_0(\phi_j r)\,\mathrm{d}r}{b^2(J_1(b\phi_j))^2}.$$

For the pressure distribution (3.1) the coefficients are

$$A_j = 3\frac{\sin\left(cP^{\frac{1}{3}}\phi_j\right) - c\phi_j P^{\frac{1}{3}}\cos\left(cP^{\frac{1}{3}}\phi_j\right)}{\pi b^2 c^3 \phi_j^3 (J_1(b\phi_j))^2} \tag{3.3}$$

with

$$c \equiv \left(\frac{3R(2\mu+\lambda)}{16\mu(\mu+\lambda)}\right)^{\frac{1}{3}}.$$

For any details concerning the convergence of the expansion (3.2) we refer to p. 37 of Sneddon (1966).

The boundary conditions on the stress components (2.4) yield the following linear system

$$\begin{aligned}
&\left(2\phi_j^2\mu - \rho\omega^2\right)\left(C_1^{(j)}\cosh(\alpha_j h) + C_2^{(j)}\sinh(\alpha_j h)\right)\\
&\quad + 2\phi_j^2\mu\left(C_3^{(j)}\cosh(\beta_j h) + C_4^{(j)}\sinh(\beta_j h)\right) = 0,\\
&\left(2\phi_j^2\mu - \rho\omega^2\right)\left(C_1^{(j)}\cosh(\alpha_j h) - C_2^{(j)}\sinh(\alpha_j h)\right)\\
&\quad + 2\phi_j^2\mu\left(C_3^{(j)}\cosh(\beta_j h) - C_4^{(j)}\sinh(\beta_j h)\right) = 0,\\
&2\mu\beta_j\alpha_j\left(C_1^{(j)}\sinh(\alpha_j h) - C_2^{(j)}\cosh(\alpha_j h)\right)\\
&\quad + \left(2\phi_j^2\mu - \rho\omega^2\right)\left(C_3^{(j)}\sinh(\beta_j h) - C_4^{(j)}\cosh(\beta_j h)\right) = 0,\\
&2\mu\beta_j\alpha_j\left(C_1^{(j)}\sinh(\alpha_j h) + C_2^{(j)}\cosh(\alpha_j h)\right)\\
&\quad + \left(2\phi_j^2\mu - \rho\omega^2\right)\left(C_3^{(j)}\sinh(\beta_j h) + C_4^{(j)}\cosh(\beta_j h)\right) - A_j\beta_j = 0,
\end{aligned} \tag{3.4}$$

which gives the coefficients

$$C_1^{(j)} = A_j\Omega_1^{(j)},\quad C_2^{(j)} = A_j\Omega_2^{(j)},\quad C_3^{(j)} = A_j\Omega_3^{(j)},\quad C_4^{(j)} = A_j\Omega_4^{(j)}, \tag{3.5}$$

where

$$\Omega_1^{(j)} = \frac{2\mu\phi_j^2\beta_j(1+\mathrm{e}^{2\beta_j h})\mathrm{e}^{\alpha_j h}}{D_1^{(j)}},\quad \Omega_2^{(j)} = \frac{2\mu\phi_j^2\beta_j(1-\mathrm{e}^{2\beta_j h})\mathrm{e}^{\alpha_j h}}{D_2^{(j)}},$$

$$\Omega_3^{(j)} = \frac{\beta_j\left(\rho\omega^2 - 2\mu\phi_j^2\right)(1+\mathrm{e}^{2\alpha_j h})\mathrm{e}^{\beta_j h}}{D_3^{(j)}},\quad \Omega_4^{(j)} = \frac{\beta_j\left(\rho\omega^2 - 2\mu\phi_j^2\right)(1-\mathrm{e}^{2\alpha_j h})\mathrm{e}^{\beta_j h}}{D_4^{(j)}},$$

and

$$D_1^{(j)} = D_3^{(j)} = \left(\rho^2\omega^4 + 4\phi_j^2\mu\left(\phi_j^2\mu - \rho\omega^2\right)\right)\left(1+\mathrm{e}^{2\alpha_j h}\right)\left(1-\mathrm{e}^{2\beta_j h}\right) - 4\phi_j^2\mu^2\alpha_j\beta_j\left(1-\mathrm{e}^{2\alpha_j h}\right)\left(1+\mathrm{e}^{2\beta_j h}\right),$$

$$D_2^{(j)} = D_4^{(j)} = -\left(\rho^2\omega^4 + 4\phi_j^2\mu\left(\phi_j^2\mu - \rho\omega^2\right)\right)\left(1-\mathrm{e}^{2\alpha_j h}\right)\left(1+\mathrm{e}^{2\beta_j h}\right) + 4\phi_j^2\mu^2\alpha_j\beta_j\left(1+\mathrm{e}^{2\alpha_j h}\right)\left(1-\mathrm{e}^{2\beta_j h}\right).$$

The displacement field is obtained by substituting the coefficients (3.5) in (2.14) and the stress components explicitly written, become

$$\begin{aligned}
\sigma_{rr}^{(j)} &= \frac{A_j}{\phi_j r \beta_j} e^{i\omega t}\Big(2\mu\alpha_j\beta_j(J_1(\phi_j r) - r\phi_j J_0(\phi_j r))\Big(\Omega_1^{(j)}\sinh(\alpha_j z) + \Omega_2^{(j)}\cosh(\alpha_j z)\Big) \\
&\quad + \Big(2\mu\phi_j^2 J_1(\phi_j r) - \Big((2\mu+\lambda)\phi_j^2 - \lambda\beta_j^2\Big) r\phi_j J_0(\phi_j r)\Big)\Big(\Omega_3^{(j)}\sinh(\beta_j z) + \Omega_4^{(j)}\cosh(\beta_j z)\Big)\Big), \\
\sigma_{\vartheta\vartheta}^{(j)} &= -\frac{A_j}{\phi_j r \beta_j} e^{i\omega t}\Big(2\mu\alpha_j\beta_j J_1(\phi_j r)\Big(\Omega_1^{(j)}\sinh(\alpha_j z) + \Omega_2^{(j)}\cosh(\alpha_j z)\Big) \\
&\quad + \Big(2\mu\phi_j^2 J_1(\phi_j r) - \Big(\beta_j^2 - \phi_j\Big) r\phi_j \lambda J_0(\phi_j r)\Big)\Big(\Omega_3^{(j)}\sinh(\beta_j z) + \Omega_4^{(j)}\cosh(\beta_j z)\Big)\Big), \\
\sigma_{zr}^{(j)} &= -\frac{A_j}{\phi_j} e^{i\omega t}\mu J_1(\phi_j r)\Big(\Big(\alpha_j^2 + \phi_j^2\Big)\Big(\Omega_1^{(j)}\cosh(\alpha_j z) + \Omega_2\sinh(\alpha_j z)\Big) \\
&\quad + 2\phi_j^2\Big(\Omega_3^{(j)}\cosh(\beta_j z) + \Omega_4^{(j)}\sinh(\beta_j z)\Big)\Big), \\
\sigma_{zz}^{(j)} &= \frac{A_j}{\beta_j} e^{i\omega t} J_0(\phi_j r)\Big(2\mu\alpha_j\beta_j\Big(\Omega_1^{(j)}\sinh(\alpha_j z) + \Omega_2^{(j)}\cosh(\alpha_j z)\Big) \\
&\quad + \Big((2\mu+\lambda)\beta_j^2 - \lambda\phi_j^2\Big)\Big(\Omega_3^{(j)}\sinh(\beta_j z) + \Omega_4^{(j)}\cosh(\beta_j z)\Big)\Big).
\end{aligned} \tag{3.6}$$

We notice that $\sigma_{rr}(b,z,t)$ and M_r, the bending moment on the edge of the plate, are both not vanishing. This result is not consistent with the notion of *simple support* imposed in classical two-dimensional theory of plates. However, Levinson's theory too does not require that we prescribe the stress components $\sigma_{rr}(b,z,t)$, $\sigma_{rz}(b,z,t)$ to find the solution; actually, the main assumptions (2.2) are, on one hand, restrictions on the displacement field, and, on the other hand, compatibility conditions on the data (Podio Guidugli et al., 1999).

4. Impact law in thick plate

In the circular thick plate the impact law is obtained by using Eq. (2.14b) for $z = +h, r = 0$ and the coefficients $C_\alpha^{(j)}, \alpha = 1,\ldots,4$ given by Eq. (3.5) which where, in turn, obtained by taking into account the boundary conditions (2.4). According to Levinson's theory, the indentation is

$$\delta^{D} = \sum_{j=1}^{\infty} K_j^{D}\Big(\sin\Big(cP^{\frac{1}{3}}\phi_j\Big) - cP^{\frac{1}{3}}\phi_j\cos\Big(cP^{\frac{1}{3}}\phi_j\Big)\Big)e^{i\omega t}, \tag{4.1}$$

where

$$K_j^{D} = \frac{8\mu(\mu+\lambda)\rho\omega^2\beta_j}{(2\mu+\lambda)\pi R b^2\phi_j^3(J_1(b\phi_j))^2}\left(\frac{1}{D_1^{(j)}}(1+e^{2\beta_j h})(1+e^{2\alpha_j h}) - \frac{1}{D_2^{(j)}}(1-e^{2\beta_j h})(1-e^{2\alpha_j h})\right).$$

Let us remark that, in the statical case, the indentation is

$$\delta^{S} = \sum_{j=1}^{\infty} K_j^{S}\Big(\sin\Big(cP^{\frac{1}{3}}\phi_j\Big) - cP^{\frac{1}{3}}\phi_j\cos\Big(cP^{\frac{1}{3}}\phi_j\Big)\Big), \tag{4.2}$$

where

$$K_j^{S} = \frac{8\big(1 - e^{8\phi_j h} - 8\phi_j h e^{4\phi_j h}\big)}{\pi R b^2\phi_j^4(J_1(b\phi_j))^2\big(-1 + e^{4\phi_j h} + 4\phi_j h e^{2\phi_j h}\big)\big(1 - e^{4\phi_j h} + 4\phi_j h e^{2\phi_j h}\big)}.$$

It is easy to check that $\lim_{\omega\to 0} K_j^{\rm D} = K_j^{\rm S}$; therefore, for low frequencies, the impact law is the same in both static and the dynamic cases. We notice that the existence of a periodic load may give rise to the resonance of the plate-like body. In fact, we find the frequency resonance values by maximizing the expression of the displacement in the first contact point (4.1).

It is possible to verify that the lower frequency value maximizing (4.1) furnishes the same free frequency value obtained by the homogeneous system associated with (3.4).

Finally, we observe that the accurate knowledge of the localized maximum stresses in the contact region and of the resonance peaks is useful for finding the optimal thickness of a plate in order to avoid undesired resonance phenomena of the plate without compromising the material resistance.

5. Equation of motion

In this section we study the problem of the frictionless normal low velocity impact of a rigid sphere against an elastic plate-like body; by assuming that the *period of duration* of the impact is much larger than the time employed by the elastic waves in traversing the plate after the first impact, we adopt the static contact law (4.2).

The equation of motion during the contact takes is

$$M\ddot{\delta}(t) + P(t) = 0, \tag{5.1}$$

where $M = 4/3\rho_0\pi R^3$ is the mass of the sphere and ρ_0 its material density; $\delta(t)$ is the instantaneous indentation and $P(t)$ the instantaneous resultant pressure.

The contact law is rewritten in the following form

$$\delta = \sum_{k=2}^{\infty} \gamma_k P^{\frac{2k-1}{3}} \tag{5.2}$$

with

$$\gamma_k = (-1)^k \frac{(2k-2)}{(2k-1)!} c^{2k-1} \sum_{j=1}^{\infty} \phi_j^{2k-1} K_j^{\rm S}.$$

The expression (5.2) may be inverted, for $0 < \delta < b^2/R$, as follows

$$P = \sum_{k=2}^{\infty} \xi_k \delta^{\frac{2k-1}{3}}, \tag{5.3}$$

where the coefficients ξ_k of the inverse expansion can be obtained through a straightforward extension of the technique presented in Morse and Feshbach (1953); the first three terms are

$$\xi_2 = \frac{1}{\gamma_2}, \quad \xi_3 = -\frac{\gamma_3}{\gamma_2^{\frac{8}{3}}}, \quad \xi_4 = -\frac{1}{3}\frac{3\gamma_4\gamma_2 - 5\gamma_3^2}{\gamma_2^{\frac{13}{3}}}, \ldots$$

After substitution of (5.3) in the equation of the motion (5.1), we obtain the non-linear ordinary differential equation

$$\ddot{\delta}(t) + \frac{1}{M}\sum_{k=2}^{\infty} \xi_k \delta(t)^{\frac{2k-1}{3}} = 0, \tag{5.4}$$

accompanied by the initial conditions

$$\delta(0) = 0 \quad \text{and} \quad \dot{\delta}(0) = v_0,$$

where v_0 is the velocity of the sphere before the impact.

Eq. (5.4) may be reduced to the following first-order differential equation for $0 < t < t_c$, ($2t_c$ is the contact period)

$$\dot{\delta}(t) = \sqrt{v_0^2 - \frac{3}{M}\sum_{k=2}^{\infty}\frac{\xi_k}{k+1}\delta(t)^{\frac{2(k+1)}{3}}}. \tag{5.5}$$

We linearize Eq. (5.4) and denote its solution with

$$\delta_0(t) = v_0\sqrt{\frac{M}{\xi_2}}\sin\left(\sqrt{\frac{\xi_2}{M}}\,t\right).$$

Now, we look for a solution of (5.5) as

$$\delta(t) = \delta_0(t) + \eta(t),$$

where $\eta(t)$ denotes the first correction term.

A straightforward calculation shows that $\eta(t)$ satisfies the following first order linear ordinary differential equation

$$\dot{\eta}(t) + F(t)\eta(t) + G(t) = 0, \quad \text{with } \dot{\eta}(0) = 0, \tag{5.6}$$

where we have set

$$F(t) = \frac{\delta_0(t)\left(\xi_2 + \xi_3\delta_0(t)^{\frac{2}{3}}\right)}{M\left(v_0^2 - \frac{\xi_2}{M}\delta_0(t)^2 - \frac{3\xi_3}{4M}\delta_0(t)^{\frac{8}{3}}\right)^{\frac{1}{2}}},$$

and

$$G(t) = \left(v_0^2 - \frac{\xi_2}{M}\delta_0(t)^2\right)^{\frac{1}{2}} - \left(v_0^2 - \frac{\xi_2}{M}\delta_0(t)^2 - \frac{3\xi_3}{4M}\delta_0(t)^{\frac{8}{3}}\right)^{\frac{1}{2}}.$$

The solution of (5.6) is

$$\eta(t) = \left(-\int G(t)\mathrm{e}^{\int F(t)\mathrm{d}t}\,\mathrm{d}t + C\right)\mathrm{e}^{-\int F(t)\mathrm{d}t}, \tag{5.7}$$

where C is obtained by the condition $\eta(0) = 0$.

We observe that in Eq. (5.6) the second order terms in $\eta(t)$ have been neglected. Actually, successive terms in the expansion furnish not significant increment in the valuation of the non-linear contribute; with analogous calculation one can get also the successive, higher order correction terms.

The numerical solutions of the equation of motion carry out the maximum indentation δ_c, the contact period $2t_c$ and the maximum contact force F_c.

6. Numerical results

In this section we consider a rigid sphere of radius $R = 0.01270$ m, an isotropic circular plate of radius $b = 0.038$ m and elastic properties $\lambda = 0.36$ GPa and $\mu = 0.43$ GPa.

By using the analytical solutions (4.2), in Fig. 1 we show the impact law in the case of low frequencies for different ratios h/b of the plate-like body (with a 100 terms expansion). As expected, there is an agreement with the Hertz's contact law only for high values of the ratio h/b.

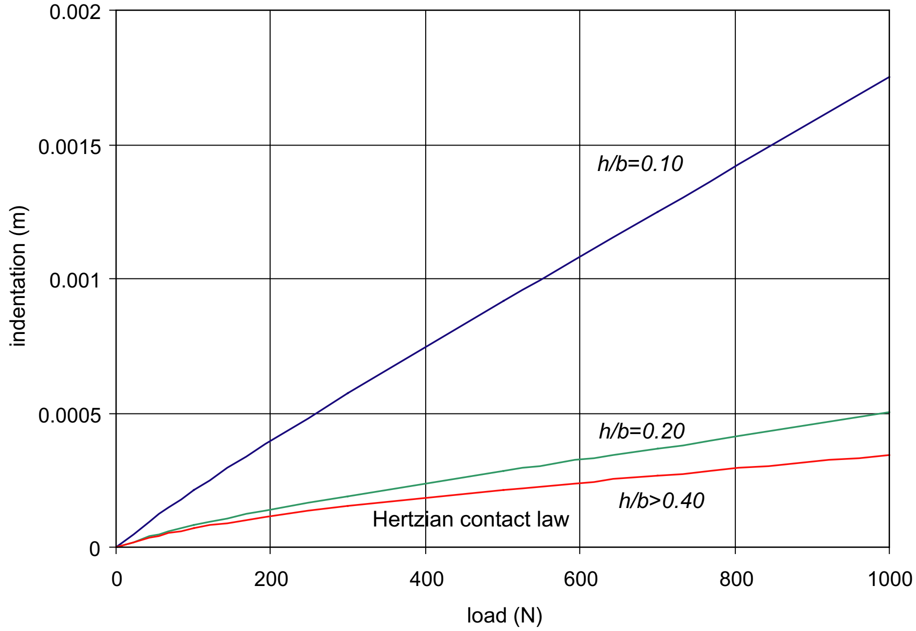

Fig. 1. Impact law for low frequencies.

In Fig. 2 we show the behavior of the indentation with respect to the frequency; we observe that the indentation (4.1) increases slowly with the increasing of the frequency when we consider values not near the

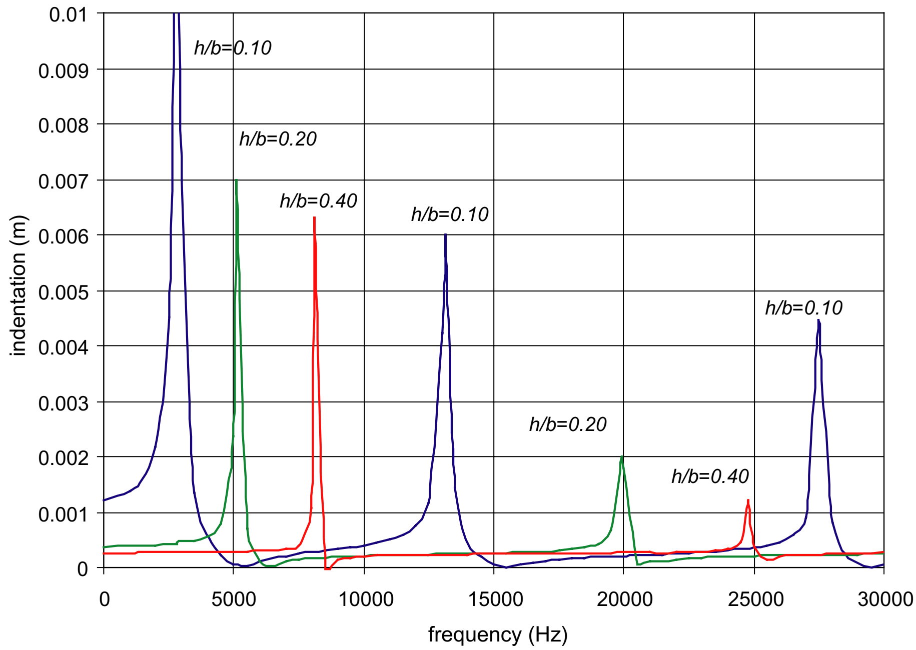

Fig. 2. Resonance frequencies for different ratios h/b.

resonance peaks; moreover, for a prescribed frequency value, an increase of the thickness produces a decrease of the indentation.

We notice that the results obtained in Section 4 give a method to optimize the thickness of a plate by taking into account the working frequency and the properties of the material. First of all we observe that a numerical analysis shows that only σ_{rr} depends effectively on the ratio h/b. In Fig. 3, we plot the "maximum" value of the radial stress with respect to h/b, obtained by using expression (3.6a) for $r = 0, z = h/2$ and for ω the first resonance value up to one half of the amplitude of the corresponding resonance peak. For different ratios h/b, the maximum radial stresses decreases with the increase of the thickness, while the resonance frequencies, obtained maximizing the (4.1), increase with the thickness. Thus, an optimization criteria may be so formulated: if we design a plate with prescribed elastic moduli and assigned working frequencies, the points of the curve of Fig. 3 yield the optimal thickness before that resonance peak and material failures arise.

Finally, we study a numerical solution of the motion equation (5.4) obtained by using the perturbative method presented in Section 5. We assume that the sphere is made of iron with density $\rho_0 = 8000$ kg/mc, that the initial velocity $v_0 = 1$ m/s and the thickness-to plate radius ratio $h/b = 0.10$. We remark that only the first correction term obtained by solution (5.7) is considered, since successive terms give not significant contribution to the solution.

The numerical evaluation gives the following results: the maximum indentation is $\delta_c = 3.6 \times 10^{-4}$ m (with $\delta_0 = 3.668 \times 10^{-4}$ m and $\eta = -3.5 \times 10^{-6}$ m), the duration of the impact is $2t_c = 1.15 \times 10^{-3}$ s and the maximum contact force is $F_c = 191$ N. Further, we compare this results with the numerical values derived by the classic Hertz's theory: ($\delta_c^{\text{hertz}} = 1.9 \times 10^{-4}$ m, $2t_c^{\text{hertz}} = 5.6 \times 10^{-4}$ s, $F_c^{\text{hertz}} = 448$ N). In Fig. 4 we show how the time of collision decreases with the increasing of the thickness-to plate radius ratio; analogously, under the same assumption, the maximum indentation δ_c decreases and the maximum contact force F_c increases. All these values approaches the Hertz's values when the plate thickness increases.

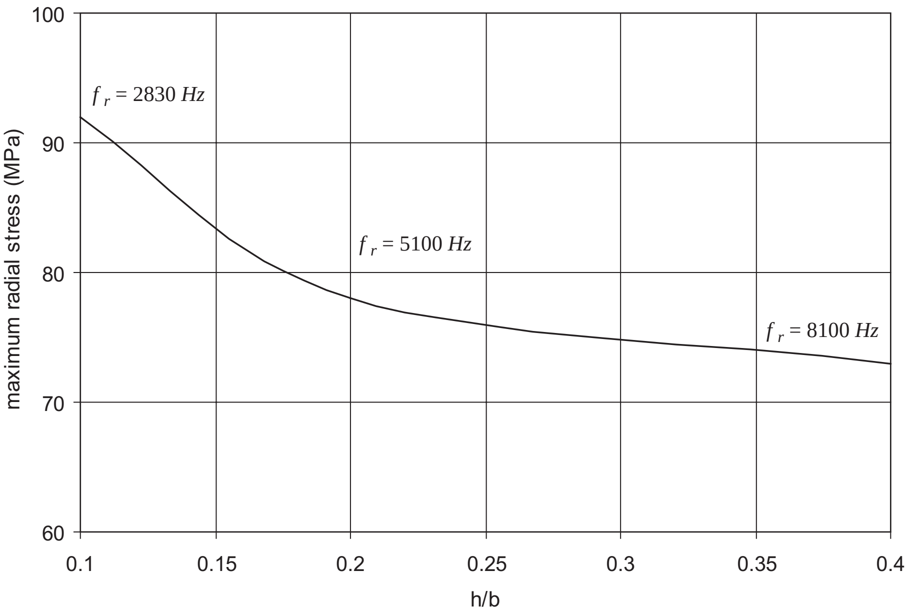

Fig. 3. Maximum radial stress for different ratios h/b at the start up of the resonance peak.

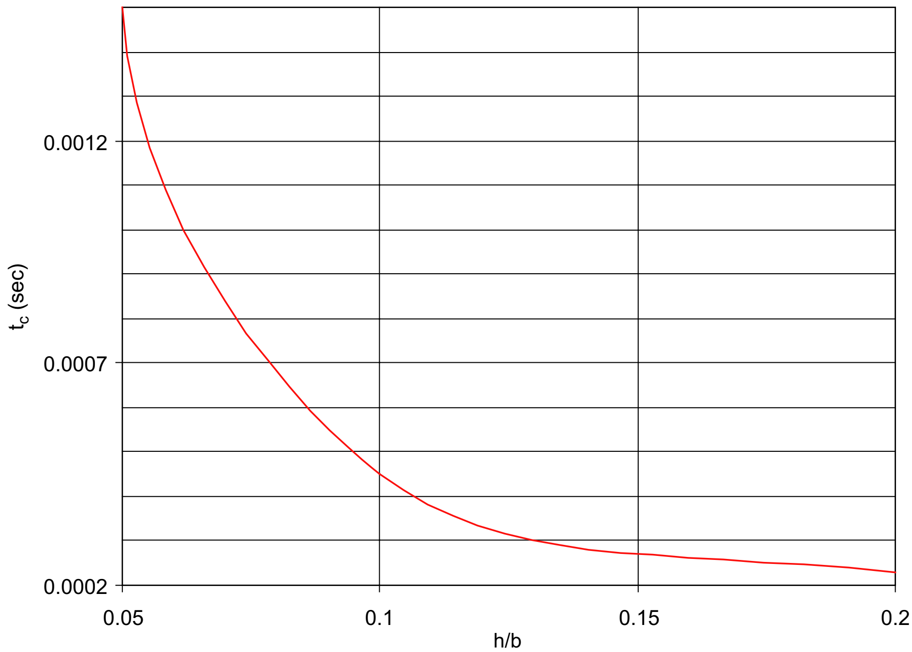

Fig. 4. Time of the collision.

7. Conclusions

A three-dimensional axisymmetrical solution for the normal impact problem of a rigid sphere impinging against an elastic plate-like body was found in order to give an impact law taking into account the thickness of the plate. It was found how the impact law deviates significantly from the Hertz's contact law in thin plate.

The equation of the motion is solved under the assumption that the static contact law applies and its solution gives the duration of the impact, the maximum indentation and pressure. Comparisons with the Hertz's theory numerical results reveal that the deformability of plate leads to a reduction of the duration of the impact in thin plates.

We remark that the solution here presented is valid only when $(a/2h < 1)$ (Chen and Frederick, 1993); if this assumption is not satisfied, the contact pressure distribution deviates significantly from the Hertzian prediction and this effect can be attributed to the tendency of the plate to wrap around the sphere (Keer and Miller, 1983).

References

Chen, C.F., Frederick, D., 1993. Contact of isotropic square plates with rigid spherical indenters. International Journal of Solids and Structures 30, 637–650.

Courant, R., Hilbert, D., 1953. Methods of Mathematical Physics. Interscience Publishers, New York.

Johnson, K.L., 1985. Contact Mechanics. Cambridge University Press.

Keer, L.M., Miller, G.R., 1983. Contact between an elastically supported circular plate and a rigid indenter. International Journal of Engineering Science 6, 681–690.

Levinson, M., 1985. The simply supported rectangular plate: an exact, three-dimensional, linear elastic solution. Journal of Elasticity 7, 283–291.

Morse, P.M., Feshbach, H., 1953. Methods of Theoretical Physics. McGraw-Hill Press.
Podio Guidugli, P., Nicotra, V., Tiero, A., 1999. Exact equilibrium solutions for linearly elastic plate-like bodies. Journal of Elasticity 56, 231–245.
Sneddon, I.N., 1966. Mixed Boundary Value Problems in Potential Theory. John Wiley and Sons, New York, USA.
Villaggio, P., 1996. The rebound of an elastic sphere against a rigid wall. Journal of Applied Mechanics 63, 259–263.
Villaggio, P., 1997. Mathematical Models for Elastic Structures. Cambridge University Press.
Watson, G.N., 1966. A Treatise on the Theory of Bessel Functions. Cambridge University Press.